

Thermodynamic Interpretation of the Field Equations of BTZ Charged Black Hole near the Horizon

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As is already known, a spacetime horizon acts like a boundary of a thermal system and we can associate with it notions as temperature and entropy. Following the work of M. Akbar, in this paper we will show how it is possible to interpret the field equation of a charged BTZ black hole near horizon as a thermodynamic identity $dE = TdS + P_r dA + \Phi dQ$, where Φ is the electric potential and Q is the electric charge of BTZ black hole. These results indicate that the field equations for the charged BTZ black hole possess intrinsic thermodynamic properties near horizon.

I. INTRODUCTION

The event horizon of a black hole acts as the boundary of the spacetime because it blocks any physical information to flow out from the black hole to the rest of world. Bekenstein [7] showed that the black holes has non-zero entropy (because they withhold information), while Hawking [3] showed that black holes emit thermal radiation with a temperature proportional to its surface gravity at the black hole horizon

$$T = \frac{\kappa}{2\pi} \quad (1)$$

and with an entropy proportional to its horizon area [8],

$$S = \frac{A}{4G}. \quad (2)$$

These quantities are connected through the identity

$$dE = TdS, \quad (3)$$

that is called *first law of black hole thermodynamics* [3, 7, 8]. When the black hole has other properties as angular momentum J and electric charge Q (e.g in the Kerr-Newman solution), the first law can be generalized to

$$dE = TdS + \Omega dJ + \Phi dQ, \quad (4)$$

where $\Omega = \frac{\partial M}{\partial J}$ is the angular velocity and $\Phi = \frac{\partial M}{\partial Q}$ is the electric potential.

The first law indicates that it could be possible to obtain a thermodynamic interpretation of the Einstein field equations near horizon [10], since the geometric quantities of the spacetime are

related with the thermodynamic quantities. This fact was used by Jacobson [4] to find Einstein equations using the first law and the proportionality between entropy and horizon area. On the other side, Paranjape et.al. [9] made an interpretation of the field equations as a thermodynamic law $TdS = dE + PdV$ near the horizon of a special class of spherically symmetric black hole. Kothawala et.al. [12] extended this kind of interpretation to stationary axis-symmetric horizons using the virtual displacement of the horizon. This approach was used very recently by M. Akbar [1] to make a thermodynamical interpretation of the field equations of (2+1) gravity near the horizon of the BTZ black hole.

In this paper we will use the same method to obtain a thermodynamical identity using the field equation and considering the virtual displacement of the horizon of a charged BTZ black hole.

II. THE CHARGED BTZ BLACK HOLE

The (2+1)-dimensional BTZ (Banados-Teitelboim-Zanelli) black holes have obtained a great importance in recent years because they provide a simplified model for exploring some conceptual issues, not only about black hole thermodynamics [5, 6] but also about quantum gravity and string theory.

The charged BTZ black hole is a solution of the (2+1)-dimensional gravity theory with a negative cosmological constant $\Lambda = -1/\ell^2$. The metric is given by [2]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\varphi^2, \quad (5)$$

where

$$f(r) = -M + \frac{r^2}{\ell^2} + \frac{Q^2}{2} \ln \left[\frac{r}{\ell} \right] \quad (6)$$

is known as the lapse function and M and Q are the mass and electric charge of the BTZ black hole, respectively. The electric potential of this black hole is

$$\Phi = \frac{\partial M}{\partial Q} = -Q \ln \left[\frac{r}{\ell} \right]. \quad (7)$$

As it can be seen, the lapse function vanishes at the radii $r = r_{\pm}$, where r_+ gives the position of the event horizon. The Bekenstein-Hawking entropy of the BTZ black holes is twice the perimeter of their event horizon [11]

$$S = 4\pi r_+, \quad (8)$$

while the Hawking temperature is given, as usual, by

$$T = \frac{1}{4\pi} \left| \frac{df(r)}{dr} \right|_{r=r_+}. \quad (9)$$

These quantities obey the first law of thermodynamics, and we will show, following the work of M. Akbar[1], that the field equations have the thermodynamic interpretation of a first law near the horizon. In order to obtain this interpretation, we will assume that the function $f(r)$ has a zero at $r = r_+$ and $f'(r_+) \neq 0$ but has a finite value at $r = r_+$. These conditions assure that we have a space-time horizon at $r = r_+$ and we can associate a non-zero surface gravity $\kappa = \frac{1}{2}f'(r_+)$ and a temperature $T = \kappa/2\pi$, while the associated entropy will be proportional to the horizon area.

III. FIELD EQUATIONS AS A THERMODYNAMIC IDENTITY

In $(2+1)$ -dimensional gravity, the components of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ for the metric (5) are simply

$$G_t^t = G_r^r = \frac{f'(r)}{2r}, \quad (10)$$

and

$$G_\varphi^\varphi = \frac{f''(r)}{2}, \quad (11)$$

where prime stands for the derivative with respect to r . It is clear that G_0^0 and G_1^1 are equal and in particular, at the horizon $r = r_+$, we have

$$G_0^0|_{r=r_+} = G_1^1|_{r=r_+} = \frac{f'(r_+)}{2r_+}. \quad (12)$$

The $(2+1)$ -dimensional field equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\pi T_{\mu\nu}, \quad (13)$$

where the units are such that $G = \frac{1}{8}$ and $c = 1$. Here $T_{\mu\nu}$ is the stress-energy tensor, and it is such that $T_r^r = P_r$, with P_r the radial pressure. The (r, r) component of the field equations for this metric, when evaluated at $r = r_+$, is

$$\frac{1}{2a}f'(r_+) - \frac{1}{\ell^2} = -\pi T_r^r = -\pi P_r \quad (14)$$

Now, we consider a virtual displacement dr_+ of the horizon and we multiply both sides of this equation by it,

$$\frac{f'(r_+)}{4\pi}d(4\pi r_+) - \frac{d(r_+^2)}{\ell^2} = -P_r d(\pi r_+^2). \quad (15)$$

The term $\frac{f'(a)}{4\pi}$ on the LHS is the associated temperature T while the accompanying quantity inside parentheses is the entropy S associated with the horizon. Using the condition $f(r_+) = 0$ we have

$$-M + \frac{r_+^2}{\ell^2} + \frac{Q^2}{2} \ln \left[\frac{r_+}{\ell} \right] = 0, \quad (16)$$

and then we obtain

$$-dM + d \left(\frac{r_+^2}{\ell^2} \right) + \frac{Q^2}{2r_+} dr_+ = 0, \quad (17)$$

$$d \left(\frac{r_+^2}{\ell^2} \right) = dM - \frac{Q^2}{2r_+} dr_+, \quad (18)$$

that corresponds to the second term in the LHS of equation (15). Thus, we have the equation

$$dM = TdS + P_r dA + \frac{Q^2}{2r_+} dr_+, \quad (19)$$

Here dA is the change in horizon area and then, the term $P_r dA$ corresponds to work done against the pressure. Using the electric potential given by (7) we can write

$$\frac{Q}{r_+} dr_+ = -dQ \ln \left[\frac{r_+}{\ell} \right], \quad (20)$$

so the field equation is

$$dM = TdS + P_r dA - \frac{Q}{2} \ln \left[\frac{r_+}{\ell} \right] dQ \quad (21)$$

$$dM = TdS + P_r dA + \frac{\Phi}{2} dQ. \quad (22)$$

As we can see, this equation resembles almost perfectly the first law of thermodynamics but there is one more step left in order to cancel the $\frac{1}{2}$ factor in the last term. Following the appreciation of Martinez et.al.[2] we can add on both sides of this equation the term $-\frac{1}{2}Q \ln \left[\frac{r_+}{\ell} \right] dQ = \frac{1}{2}\Phi dQ$, that is just the electrostatic energy outside a sphere of radius r_+ (in our case this sphere is the horizon of the black hole). Thus, equation (22) is

$$dE = TdS + P_r dA + \Phi dQ, \quad (23)$$

where E is given by

$$dE = dM - \frac{1}{2}Q \ln \left[\frac{r_+}{\ell} \right] dQ, \quad (24)$$

and can be thought as the total energy within the radius r_+ . Hence, the field equation near horizon of the charged BTZ black hole can be expressed as the thermodynamic identity²³ under the virtual displacement of the horizon.

IV. CONCLUSION

We have shown that the field equations for the charged BTZ black hole have a thermodynamic interpretation near the horizon. As it has been shown, we obtained the first law $dE = TdS + P_r dA + \Phi dQ$, where P_r is the radial pressure of the source, A is the area enclosed by the horizon, Φ is the electric potential, Q is the electric charge and E is the total energy inside the horizon of the charged BTZ black hole, that is the mass of the black hole minus a term that corresponds to the electrostatic energy outside the horizon. The charged rotating BTZ black hole case can be considered for further investigation.

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- [1] M. Akbar. *Thermodynamic Interpretation of Field Equations at Horizon of BTZ Black Hole*. Chin. Phys. Lett. **24**, **5**. (2007)1158.
 - [2] C. Martinez, C. Teitelboim and J. Zanelli. *Charged Rotating Black Hole in Three Spacetime Dimensions*. Phys. Rev. **D61** (2000) 104013 **hep-th/9912259**
 - [3] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
 - [4] T. Jacobson, Phys. Rev. Lett. **75**(1995)1260.
 - [5] S. Carlip, Class. Quant. Grav. **12**(1995)2853.
 - [6] A. Ashtekar, Adv. Theor. Math. Phys. **6**(2002)507.
 - [7] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973); Phys. Rev. **D9**, 3293 (1974).
 - [8] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31**, 161 (1973).
 - [9] A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. **D74**, 104015(2006) **hep-th/0607240**.
 - [10] T. Padmanabhan, Mod. Phys. Letts. **A17**, 923 (2002) **gr-qc/0202078**. Phys. Rept. **406**, 49 (2005) **gr-qc/0311036**.
 - [11] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69**, 1849(1992); M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D **48**, 1506(1993).
 - [12] D. kothawala, S. Sarkar, and T. Padmanabhan, **gr-qc/0701002**.